

A Criterion for Irreducibility of Parabolic Baby Verma Modules of Reductive Lie Algebras

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ABSTRACT. Let G be a connected, reductive algebraic group over an algebraically closed field k of prime characteristic p and $\mathfrak{g} = \text{Lie}(G)$. In this paper, we study representations of \mathfrak{g} with a p -character χ of standard Levi form. When \mathfrak{g} is of type A_n, B_n, C_n or D_n , a sufficient condition for the irreducibility of standard parabolic baby Verma \mathfrak{g} -modules is obtained. This partially answers a question raised by Friedlander and Parshall in [Friedlander E. M. and Parshall B. J., *Deformations of Lie algebra representations*, Amer. J. Math. 112 (1990), 375-395]. Moreover, as an application, in the special case that \mathfrak{g} is of type A_n or B_n , and χ lies in the sub-regular nilpotent orbit, we recover a result of Jantzen in [Jantzen J. C., *Subregular nilpotent representations of sl_n and so_{2n+1}* , Math. Proc. Cambridge Philos. Soc. 126 (1999), 223-257].

1. Introduction and main results

The modular representations of reductive Lie algebras in prime characteristic have been developed over the past decades with intimate connections to algebraic groups (cf. [7], [1, 2], [3, 4], [10], [8], [11] etc.).

Let k be an algebraically closed field of prime characteristic p and G be a connected, reductive algebraic group over k with $\mathfrak{g} = \text{Lie}(G)$. Fix a maximal torus T of G and let $X(T)$ be the character group of T . Assume that the derived group $G^{(1)}$ of G is simply connected, p is a good prime for the root system of \mathfrak{g} , and \mathfrak{g} has a non-degenerated G -invariant bilinear form. Associated with any given linear form $\chi \in \mathfrak{g}^*$, the χ -reduced enveloping algebra $U_\chi(\mathfrak{g})$ is defined to be the quotient of the universal enveloping algebra $U(\mathfrak{g})$ by the ideal generated by all $x^p - x^{[p]} - \chi(x)^p$ with $x \in \mathfrak{g}$. Each isomorphism class of irreducible representations of \mathfrak{g} corresponds to a

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unique p -character χ . Furthermore, a well-known result of Kac-Weisfeiler shows that there is a Morita equivalence between $U_\chi(\mathfrak{g})$ -module category and $U_\chi(\mathfrak{l})$ -module category, where \mathfrak{l} is a certain reductive subalgebra of \mathfrak{g} such that $\chi|_{[\mathfrak{l}, \mathfrak{l}]}$ is nilpotent (cf. [7] and [1]). This enables us to study representations of $U_\chi(\mathfrak{g})$ just with nilpotent χ .

We say a p -character χ has standard Levi form if χ is nilpotent and if there exists a subset I of Π such that $\chi(\mathfrak{g}_{-\alpha}) \neq 0$ for $\alpha \in I$ and $\chi(\mathfrak{g}_{-\alpha}) = 0$ for $\alpha \in R^+ \setminus I$, where Π is the set of all simple roots, R^+ is the set of all positive roots and $|R^+| = N$ (cf. [3, §10]). In this paper, we study representations of \mathfrak{g} with a p -character χ of standard Levi form. Fix a triangular decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ and let $\mathfrak{b}^+ = \mathfrak{h} \oplus \mathfrak{n}^+$. For a subset I in Π , set $J = \Pi \setminus I$. Let $\mathfrak{g}_J = \mathfrak{n}_J^- \oplus \mathfrak{h} \oplus \mathfrak{n}_J^+$ and $\mathfrak{p}_J = \mathfrak{g}_J \oplus \mathfrak{u}_J^+$ be the Levi subalgebra and the parabolic subalgebra of \mathfrak{g} corresponding to J , respectively. Denote by $\widehat{L}_{\mathfrak{g}_J}(\lambda)$ the irreducible $X(T)/\mathbb{Z}I$ -graded $U_\chi(\mathfrak{g}_J)$ -module with “highest” weight λ (note that $U_\chi(\mathfrak{g}_J) = U_0(\mathfrak{g}_J)$). Then $\widehat{L}_{\mathfrak{g}_J}(\lambda)$ can be extended to a $U_\chi(\mathfrak{p}_J)$ -module with trivial \mathfrak{u}_J^+ -action. The induced module $U_\chi(\mathfrak{g}) \otimes_{U_\chi(\mathfrak{p}_J)} \widehat{L}_{\mathfrak{p}_J}(\lambda)$ is called a parabolic baby Verma module, denoted by $\widehat{\mathcal{Z}}_P(\lambda)$.

In [2, § 5.1], Friedlander and Parshall put forward the following open question:

QUESTION 1.1. Can one give necessary and sufficient conditions on an irreducible module for a parabolic subalgebra \mathfrak{p}_J to remain irreducible upon induction to \mathfrak{g} ?

When χ is regular nilpotent, Friedlander-Parshall answered this question in [1]. They showed that all such inductions remain irreducible. When \mathfrak{g} is of type A_2 , and $\chi(\neq 0)$ is of standard Levi form, then each irreducible \mathfrak{g} -module with a p -character χ is a parabolic baby Verma module. Quite recently, the authors of the present paper obtained a necessary and sufficient condition for irreducibility of parabolic baby Verma modules of $\mathfrak{sl}(4, k)$ in [9].

Let $C_0 = \{\lambda \in X(T)_{\mathbb{R}} \mid 0 \leq \langle \lambda + \rho, \alpha^\vee \rangle < p \text{ for all } \alpha \in R^+\}$ be the first dominant alcove of $X(T)_{\mathbb{R}}$. Let $X_1(T) = \{\lambda \in X(T) \mid 0 \leq \langle \lambda + \rho, \alpha^\vee \rangle < p \text{ for all } \alpha \in \Pi\}$ and $X'_1(T) \subset X_1(T)$ be a system of representatives for $X(T)/pX(T)$. Each $\lambda \in X(T)$ has a unique decomposition $\lambda = \lambda_0 + p\lambda_1$ with $\lambda_0 \in X'_1(T)$ and $\lambda_1 \in X(T)$ (cf. [5, II § 9.14]). For each $\lambda \in X(T)$, the map $\lambda \mapsto d\lambda$ induces a bijection $X(T)/pX(T) \cong \Lambda = \{\mu \in \mathfrak{h}^* \mid \mu(h)^p - \mu(h^{[p]}) = 0, \forall h \in \mathfrak{h}\}$ (cf. [3, § 11.1]). So we can regard $X'_1(T)$ as a system of representatives for Λ . We call $\lambda \in X(T)$ p -regular if the stabilizer of λ in W_p is trivial, where W_p is the affine Weyl group of \mathfrak{g} .

In the present paper, we give a sufficient condition on the irreducibility of some parabolic baby Verma modules, which partially answers Question 1.1.

THEOREM 1.2. *Let \mathfrak{g} be of type A_n, B_n, C_n or D_n satisfying the hypotheses (H1)-(H3) in Section 2.1. Let $\lambda = \lambda_0 + p\lambda_1 \in X(T)$ such that $\lambda_0 \in C_0$ and $\lambda_1 \in X(T)$. Let $\chi \in \mathfrak{g}^*$ be of standard Levi form and $I = \{\alpha \in \Pi \mid \chi(\mathfrak{g}_{-\alpha}) \neq 0\}$ in the Dynkin*

diagram of \mathfrak{g} is one of the following forms

$$\begin{cases} \text{(i) or (ii),} & \text{for } \mathfrak{g} = A_n, \\ \text{(iii),} & \text{for } \mathfrak{g} = B_n, \\ \text{(iv),} & \text{for } \mathfrak{g} = C_n, \\ \text{(v) or (vi),} & \text{for } \mathfrak{g} = D_n, \end{cases}$$

where

$$\text{(i)} \quad \underbrace{\bullet - \dots - \bullet - \bullet}_I - \circ - \dots - \circ,$$

$$\text{(ii)} \quad \circ - \circ - \dots - \underbrace{\bullet - \dots - \bullet - \bullet}_I.$$

$$\text{(iii)} \quad \circ - \dots - \circ - \underbrace{\bullet - \dots - \bullet \Rightarrow \bullet}_I.$$

$$\text{(iv)} \quad \underbrace{\bullet - \bullet - \dots - \bullet}_I - \circ - \dots - \circ \Leftarrow \circ.$$

$$\text{(v)} \quad \begin{array}{ccccccc} \circ & \dots & - \circ & - & \bullet & - & \dots & - & \bullet & - & \bullet \\ & & & & & & & & | & & \\ & & & & & & & & \bullet & & \end{array}$$

$$\text{(vi)} \quad \begin{array}{ccccccc} \bullet & - & \dots & - & \bullet & - & \dots & - & \bullet & - & \bullet \\ & & & & & & & & | & & \\ & & & & & & & & \circ & & \end{array}$$

Set $J = \Pi \setminus I$. Then the parabolic baby Verma module $\widehat{\mathcal{Z}}_P(\lambda)$ is irreducible, provided that λ is p -regular.

As a consequence of Theorem 1.2, we have

COROLLARY 1.3. Maintain the notations as in Theorem 1.2. The following statements hold.

- (1) Assume that \mathfrak{g} is of type A_n , and that $\chi \in \mathfrak{g}^*$ is sub-regular nilpotent and has standard Levi form. Then each irreducible \mathfrak{g} -module is a parabolic baby Verma module.

- (2) Assume that \mathfrak{g} is of type B_n , and that $\chi \in \mathfrak{g}^*$ is sub-regular nilpotent and has standard Levi form. Let $\{\widehat{L}_\chi(\lambda_i) \mid 1 \leq i \leq 2n\}$ be the set of isomorphism classes of simple \mathfrak{g} -modules in the block containing $\widehat{L}_\chi(\lambda_1)$ described as in [6, Proposition 3.13]. Then $\widehat{Z}_P(\lambda_i)$, $i \neq n, 2n$, is irreducible with dimension $r_i p^{N-1}$ for $1 \leq i \leq n-1$ and $r_{2n-i} p^{N-1}$ for $n+1 \leq i \leq 2n-1$.

REMARK 1.4. Corollary 1.3 coincides with the results by Jantzen in [6, Theorem 2.6, Proposition 3.13].

2. Preliminaries

2.1. Notations and assumptions. Throughout this paper, we always assume that k is an algebraically closed field of prime characteristic p . We use notations in [3].

Let G be a connected, reductive algebraic group over k and $\mathfrak{g} = \text{Lie}(G)$. Then \mathfrak{g} carries a natural restricted mapping $[p]: x \mapsto x^{[p]}$. We assume that the following three hypotheses are satisfied ([3, § 6.3]):

- (H1) The derived group $\mathcal{D}G$ of G is simply connected;
- (H2) The prime p is good for \mathfrak{g} ;
- (H3) There exists a G -invariant non-degenerate bilinear form on \mathfrak{g} .

Let T be a maximal torus of G and $X(T)$ be the character group of T . Denote respectively by R^\pm the sets of all positive roots and all negative roots. For each $\alpha \in R$, let \mathfrak{g}_α denote the root subspace of \mathfrak{g} corresponding to α and $\mathfrak{n}^+ = \sum_{\alpha \in R^+} \mathfrak{g}_\alpha$, $\mathfrak{n}^- = \sum_{\alpha \in R^-} \mathfrak{g}_\alpha$. We have the triangular decomposition: $\mathfrak{g} = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^-$ with \mathfrak{h} being the Cartan subalgebra of \mathfrak{g} with rank l . Take a Chevalley basis $\{x_\alpha, y_\alpha, h_i \mid \alpha \in R^+, 1 \leq i \leq l\}$ of \mathfrak{g} . Let $\mathfrak{b}^+ = \mathfrak{h} \oplus \mathfrak{n}^+$ be the Borel subalgebra of \mathfrak{g} . For each $\alpha \in R$, let α^\vee denote the coroot of α , W the Weyl group generated by all s_α with $\alpha \in R$, and W_p the affine Weyl group generated by $s_{\alpha, r}$ ($r \in \mathbb{Z}$), where $s_{\alpha, r}$ is the affine reflection defined by $s_{\alpha, r}(\mu) = \mu - (\langle \mu, \alpha^\vee \rangle - rp)\alpha$ for any $\mu \in \mathfrak{h}^*$. Define the dot action of w on λ by $w \cdot \lambda = w(\lambda + \rho) - \rho$ for $w \in W$ and $\lambda \in \mathfrak{h}^*$, where ρ is half the sum of all positive roots.

2.2. Baby Verma modules. Modulo Morita equivalence of representations, we can assume that $\chi(\mathfrak{b}^+) = 0$ without loss of generality (cf. [7, 1]). Set $\Lambda := \{\lambda \in \mathfrak{h}^* \mid \lambda(h)^p = \lambda(h^{[p]})\}$. Any simple $U_0(\mathfrak{h})$ -module corresponds to a unique $\lambda \in \Lambda$ (cf. [3]) and is one-dimensional, denoted by $k_\lambda = kv_\lambda$, with $h \cdot v_\lambda = \lambda(h)v_\lambda$ for any $h \in \mathfrak{h}$. Since k_λ can be extended to a $U_0(\mathfrak{b}^+)$ -module with trivial \mathfrak{n}^+ -action, we have an induced module $Z_\chi(\lambda) = U_\chi(\mathfrak{g}) \otimes_{U_0(\mathfrak{b}^+)} k_\lambda$ which is called a baby Verma module. Each simple $U_\chi(\mathfrak{g})$ -module is the homomorphic image of some baby Verma module $Z_\chi(\lambda)$, $\lambda \in \Lambda$ (cf. [3] or [4]).

2.3. Standard Levi forms. We say a p -character χ has standard Levi form if χ is nilpotent and if there exists a subset I of all simple roots such that

$$(2.1) \quad \chi(\mathfrak{g}_{-\alpha}) = \begin{cases} \neq 0, & \text{if } \alpha \in I, \\ 0, & \text{if } \alpha \in R^+ \setminus I. \end{cases}$$

As in [3, § 10.4; § 10.5], when I is the full set of all simple roots, we call χ a regular nilpotent element in \mathfrak{g}^* . When $I = \emptyset$, then $\chi = 0$. We denote by R_I the root system corresponding to the subset I , and W_I the Weyl group generated by all the s_α with $\alpha \in I$.

2.4. Graded module category. Assume that χ is of standard Levi form with $I = \{\alpha \in \Pi \mid \chi(\mathfrak{g}_{-\alpha}) \neq 0\}$. Following Jantzen [3, § 11], we can define a refined subcategory of the $U_\chi(\mathfrak{g})$ -module category, which is the $X(T)/\mathbb{Z}I$ -graded $U_\chi(\mathfrak{g})$ -module category, denoted by \mathcal{C} . For $\lambda \in X(T)$, the graded baby Verma module $\widehat{\mathcal{Z}}_\chi(\lambda)$ has a unique irreducible quotient, denoted by $\widehat{L}_\chi(\lambda)$. The latter is also a graded simple module.

2.5. Parabolic baby Verma module. Assume that χ has standard Levi form associated with a subset I of the full set Π of simple roots. Set $J = \Pi \setminus I$. Let $\mathfrak{g}_J = \mathfrak{h} \oplus \bigoplus_{\alpha \in R \cap \mathbb{Z}J} \mathfrak{g}_\alpha$, $\mathfrak{u}_J^+ = \bigoplus_{\alpha > 0, \alpha \notin \mathbb{Z}J} \mathfrak{g}_\alpha$, and $\mathfrak{p}_J = \mathfrak{g}_J \oplus \mathfrak{u}_J^+$. Then $U_\chi(\mathfrak{p}_J) = U_0(\mathfrak{p}_J)$. For $\mu \in X(T)$, let $\widehat{L}_{\mathfrak{p}_J}(\mu)$ be the graded irreducible $U_0(\mathfrak{p}_J)$ -module, which is indeed a graded irreducible $U_0(\mathfrak{g}_J)$ -module with trivial \mathfrak{u}_J^+ -action. The parabolic baby Verma module is defined as the following induced module

$$\widehat{\mathcal{Z}}_P(\lambda) := U_\chi(\mathfrak{g}) \otimes_{U_0(\mathfrak{p}_J)} \widehat{L}_{\mathfrak{p}_J}(\lambda), \lambda \in X(T).$$

Then $\widehat{\mathcal{Z}}_P(\lambda)$ is a quotient module of $\widehat{\mathcal{Z}}_\chi(\lambda)$. Let $\varphi: \widehat{\mathcal{Z}}_\chi(\lambda) \rightarrow \widehat{\mathcal{Z}}_P(\lambda)$ be the canonical surjective morphism.

2.6. We apply the translation principle to $\widehat{\mathcal{Z}}_P(\lambda)$.

LEMMA 2.1. *Let $\lambda = (0, 0, \dots, 0)$ and $\mu \in C_0 \cap X(T)$ be a regular weight. Then we have*

$$T_\lambda^\mu \widehat{\mathcal{Z}}_P(\lambda) = \widehat{\mathcal{Z}}_P(\mu)$$

where T_λ^μ is the so-called translation functor defined in [3, § 11].

PROOF. By the definition of the translation factor, we have

$$T_\lambda^\mu \widehat{\mathcal{Z}}_P(\lambda) = \text{pr}_\mu(L(\nu) \otimes \widehat{\mathcal{Z}}_P(\lambda))$$

where $L(\nu)$ is the simple G -module with highest weight ν in $W(\mu - \lambda)$ (cf. [3, § 11.20]).

Since $\widehat{L}_\chi(\lambda)$ is the head of $\widehat{\mathcal{Z}}_P(\lambda)$ and $T_\lambda^\mu \widehat{L}_\chi(\lambda) \cong \widehat{L}_\chi(\mu)$ (cf. [3, Proposition 11.21]), we have $T_\lambda^\mu \widehat{\mathcal{Z}}_P(\lambda) \neq 0$.

Let Ξ be the set of weights in $L(\nu)$. By [5, Lemma 7.7, Proposition 7.11], there exists a unique weight $\xi \in \Xi$ with $\xi + \lambda \in W_p \cdot \mu$ and $\xi + \lambda = \mu$. Note that λ is trivial, then $\xi = \mu$. Hence, μ is the unique weight of $L(\nu) \otimes k_\lambda$ lying in $W_p \cdot \mu$, where k_λ is the one-dimensional trivial $U_0(\mathfrak{g}_J)$ -module.

The generalized tensor identity (cf. [4, §1.12]) yields the following isomorphism

$$(2.2) \quad T_\lambda^\mu \widehat{\mathcal{Z}}_P(\lambda) := \text{pr}_\mu(L(\nu) \otimes \widehat{\mathcal{Z}}_P(\lambda)) \cong \text{pr}_\mu(U_\chi(\mathfrak{g}) \otimes_{U_0(\mathfrak{p}_J)} (L(\nu) \otimes \widehat{L}_{\mathfrak{p}_J}(\lambda))).$$

Since $\lambda = (0, 0, \dots, 0)$, the simple module $\widehat{L}_{\mathfrak{p}_J}(\lambda)$ is trivial, i.e., $\widehat{L}_{\mathfrak{p}_J}(\lambda) = k_\lambda$. Then $L(\nu) \otimes \widehat{L}_{\mathfrak{p}_J}(\lambda) = L(\nu) \otimes k_\lambda$. The set of weights in $L(\nu) \otimes k_\lambda$ is just Ξ .

We have the following composition series of $L(\nu) \otimes k_\lambda$

$$0 = M_0 \subset M_1 \subset M_2 \subset \dots \subset M_r = L(\nu) \otimes k_\lambda$$

where the factors $M_j/M_{j-1} \cong \widehat{L}_{\mathfrak{p}_J}(\lambda_j)$ with $\lambda_j \in \Xi$. Since $T_\lambda^\mu \widehat{\mathcal{Z}}_P(\lambda) \neq 0$ and μ is the unique weight of $L(\nu) \otimes \lambda$ which lies in $W_p \cdot \mu$, there exists a unique $l \leq r$ with $\lambda_l = \mu$. Then $\widehat{L}_{\mathfrak{p}_J}(\mu) \cong M_l/M_{l-1}$ is the unique composition factor of $L(\nu) \otimes \lambda$ whose highest weight lies in $W_p \cdot \mu$.

Since the short sequence $0 \rightarrow M_l \rightarrow M_r \rightarrow M_r/M_l \rightarrow 0$ is exact and the functor $\text{pr}_\mu(U_\chi(\mathfrak{g}) \otimes_{U_0(\mathfrak{p}_J)} (-))$ is exact, we have the following isomorphisms

$$(2.3) \quad \begin{aligned} & \text{pr}_\mu(U_\chi(\mathfrak{g}) \otimes_{U_0(\mathfrak{p}_J)} (L(\nu) \otimes \widehat{L}_{\mathfrak{p}_J}(\lambda))) \\ &= \text{pr}_\mu(U_\chi(\mathfrak{g}) \otimes_{U_0(\mathfrak{p}_J)} M_r) \\ &\cong \text{pr}_\mu(U_\chi(\mathfrak{g}) \otimes_{U_0(\mathfrak{p}_J)} M_l) \\ &\cong \text{pr}_\mu(U_\chi(\mathfrak{g}) \otimes_{U_0(\mathfrak{p}_J)} \widehat{L}_{\mathfrak{p}_J}(\mu)) \\ (2.4) \quad &\cong \text{pr}_\mu(\widehat{\mathcal{Z}}_P(\mu)). \end{aligned}$$

As $\widehat{\mathcal{Z}}_P(\mu)$ is the quotient of the baby Verma module $\widehat{Z}_\chi(\mu)$, then $\widehat{\mathcal{Z}}_P(\mu)$ has a simple head and is indecomposable. Furthermore, we have

$$(2.5) \quad \text{pr}_\mu(\widehat{\mathcal{Z}}_P(\mu)) \cong \widehat{\mathcal{Z}}_P(\mu).$$

It follows from (2.2), (2.3) and (2.5) that $T_\lambda^\mu \widehat{\mathcal{Z}}_P(\lambda) = \widehat{\mathcal{Z}}_P(\mu)$. The proof is completed. \square

Let $\widehat{M} \in \mathcal{C}$. A weight vector $m \in \widehat{M}$ is called a *maximal weight vector* if $x_\alpha \cdot m = 0$ for any $\alpha \in R^+$.

2.7. We need the following lemma for later use.

LEMMA 2.2. *Assume that $\chi(\mathfrak{b}^+) = 0$. Then every submodule of $\widehat{\mathcal{Z}}_P(\lambda)$ contains a maximal weight vector.*

PROOF. Let \widehat{M} be a submodule of $\widehat{\mathcal{Z}}_P(\lambda)$. Since $U_0(\mathfrak{b}^+)$ is a subalgebra of $U_\chi(\mathfrak{g})$, \widehat{M} is also a $U_0(\mathfrak{b}^+)$ -module. Then there exists an irreducible $U_0(\mathfrak{b}^+)$ -submodule in \widehat{M} which is one-dimensional annihilated by \mathfrak{n}^+ . This completes the proof. \square

3. Proof of Theorem 1.2 for type A_n

3.1. In this section, we always assume that \mathfrak{g} is a simple Lie algebra of type A_n and $\chi \in \mathfrak{g}^*$ has standard Levi form, and $I = \{\alpha \in \Pi \mid \chi(\mathfrak{g}_{-\alpha}) \neq 0\}$ in the Dynkin diagram is described as case (i) in Theorem 1.2.

We assume that $I = \{\alpha_1, \alpha_2, \dots, \alpha_s\}$ with $s \leq n$. Then $J = \{\alpha_{s+1}, \dots, \alpha_n\}$. If $s = n$, i.e., χ is regular nilpotent, it's well known that $\widehat{\mathcal{Z}}_P(\lambda) = \widehat{\mathcal{Z}}_\chi(\lambda)$ is irreducible (cf. [3, § 10]). When $s < n$, we have

CLAIM 3.1. The parabolic baby Verma module $\widehat{\mathcal{Z}}_P(\lambda)$ has only one maximal weight vector (up to scalars) that generate $\widehat{\mathcal{Z}}_P(\lambda)$.

REMARK 3.2. It follows from Claim 3.1 and Lemma 2.2 that $\widehat{\mathcal{Z}}_P(\lambda)$ is irreducible.

In the following subsections, we first prove Claim 3.1.

3.2. Thanks to Lemma 2.1, it suffices to prove Claim 3.1 for the special case that $\lambda = (0, 0, \dots, 0)$. In this case, the irreducible $\widehat{L}_{\mathfrak{p}_J}(\lambda)$ -module is one-dimensional. This means that this module is \mathfrak{p}_J -trivial. Then $\widehat{\mathcal{Z}}_P(\lambda) = U_\chi(\mathfrak{u}_J^-)$ as vector spaces, where \mathfrak{u}_J^- is the negative counterpart of \mathfrak{u}_J^+ such that $\mathfrak{g} = \mathfrak{u}_J^- \oplus \mathfrak{p}_J$.

3.3. In the remainder of this section, we always take $\lambda = (0, 0, \dots, 0)$. Assume that $w = u \otimes v_\lambda$ is a maximal weight vector of $\widehat{\mathcal{Z}}_P(\lambda)$ of weight μ . We aim at proving that w generates the whole $\widehat{\mathcal{Z}}_P(\lambda)$.

Suppose the submodule generated by w is proper, then $\mu \in W_p \cdot \lambda$, and $\mu = \lambda - \sum_{i=1}^n k_i \alpha_i$, $k_i \in \mathbb{Z}_+$ (cf. [4, Proposition 4.5]). Fix an order of the Chevalley basis in \mathfrak{u}_J^- as follows

$$\begin{aligned} & y_{\alpha_1}; \\ & y_{\alpha_1+\alpha_2}, y_{\alpha_2}; \\ & y_{\alpha_1+\alpha_2+\alpha_3}, y_{\alpha_2+\alpha_3}, y_{\alpha_3}; \\ & \dots \\ & y_{\alpha_1+\alpha_2+\dots+\alpha_s}, \dots, y_{\alpha_{s-1}+\alpha_s}, y_{\alpha_s}; \\ & y_{\alpha_1+\alpha_2+\dots+\alpha_{s+1}}, \dots, y_{\alpha_s+\alpha_{s+1}}; \\ & \dots \\ & y_{\alpha_1+\alpha_2+\dots+\alpha_n}, \dots, y_{\alpha_s+\dots+\alpha_n}. \end{aligned}$$

Then $U_\chi(\mathfrak{u}_J^-)$ has the following basis

$$(3.1) \quad \underline{y}_1^{\mathbf{a}_1} \underline{y}_2^{\mathbf{a}_2} \dots \underline{y}_s^{\mathbf{a}_s} \underline{y}_{s+1}^{\mathbf{a}_{s+1}} \dots \underline{y}_n^{\mathbf{a}_n},$$

where $\mathbf{a} = (\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n)$,

$$\underline{y}_j^{\mathbf{a}_j} = y_{\alpha_1+\alpha_2+\dots+\alpha_j}^{a_j(1)} y_{\alpha_2+\dots+\alpha_j}^{a_j(2)} \dots y_{\alpha_{j-1}+\alpha_j}^{a_j(j-1)} y_{\alpha_j}^{a_j(j)} \dots \text{ for } j = 1, 2, \dots, s$$

and

$$\underline{y}_j^{\mathbf{a}_j} = y_{\alpha_1+\dots+\alpha_j}^{a_j(1)} y_{\alpha_2+\dots+\alpha_j}^{a_j(2)} \cdots y_{\alpha_s+\alpha_{s+1}+\dots+\alpha_j}^{a_j(s)} \text{ for } j = s+1, \dots, n$$

with $0 \leq a_j(k) \leq p-1$, $\forall j, k$. Hence $u \otimes v_\lambda$ can be uniquely written as follows

$$(3.2) \quad u \otimes v_\lambda = \sum_{\mathbf{a}} l_{\mathbf{a}} \underline{y}_1^{\mathbf{a}_1} \underline{y}_2^{\mathbf{a}_2} \cdots \underline{y}_s^{\mathbf{a}_s} \underline{y}_{s+1}^{\mathbf{a}_{s+1}} \cdots \underline{y}_n^{\mathbf{a}_n} \otimes v_\lambda.$$

For $\widehat{M} \in \mathcal{C}$, we have a decomposition $\widehat{M} = \bigoplus_{\nu \in X(T)/\mathbb{Z}I} \widehat{M}^\nu$ with $x_\alpha \widehat{M}^\nu \subset \widehat{M}^{\nu+\alpha}$ for all $\alpha \in R$. By [3, § 11.7], we have $\widehat{Z}_\chi(\lambda)^{\lambda+\mathbb{Z}I} \cong_{\mathfrak{g}_I} \widehat{L}_\chi(\lambda)^{\lambda+\mathbb{Z}I}$. Hence, $\widehat{\mathcal{Z}}_P(\lambda)^{\lambda+\mathbb{Z}I} \cong_{\mathfrak{g}_I} \widehat{L}_\chi(\lambda)^{\lambda+\mathbb{Z}I}$. Since $u \otimes v_\lambda$ generates a proper submodule of $\widehat{\mathcal{Z}}_P(\lambda)$, it follows that

$$(3.3) \quad u \otimes v_\lambda \notin \widehat{\mathcal{Z}}_P(\lambda)^{\lambda+\mathbb{Z}I} \text{ and } \mu \in W_{p,\lambda} \setminus W_{I,p,\lambda}.$$

3.4. It follows from (3.1) and (3.2) that $k_s \geq k_{s+1}$ for a weight vector $u \otimes v_\lambda \in \widehat{\mathcal{Z}}_P(\lambda)$ of weight $\mu = \lambda - \sum_{i=1}^n k_i \alpha_i$, $k_i \in \mathbb{Z}_+$. Furthermore, we have

LEMMA 3.3. Assume that $u \otimes v_\lambda$ is a maximal weight vector of $\widehat{\mathcal{Z}}_P(\lambda)$ with weight $\mu = \lambda - \sum_{i=1}^n k_i \alpha_i$, $k_i \in \mathbb{Z}_+$, and that $u \otimes v_\lambda$ generates a proper submodule of $\widehat{\mathcal{Z}}_P(\lambda)$. Then $k_s > k_{s+1}$.

PROOF. Suppose $k_s = k_{s+1} \neq 0$. Then the factor $\underline{y}_s^{\mathbf{a}_s}$ does not appear in any monomial summand of (3.2). The expression (3.2) can be written as

$$(3.4) \quad u \otimes v_\lambda = \sum_{\mathbf{a}} l_{\mathbf{a}} \underline{y}_1^{a_1(1)} \underline{y}_2^{(a_2(1), a_2(2))} \cdots \underline{y}_{s-1}^{(a_{s-1}(1), a_{s-1}(2), \dots, a_{s-1}(s-1))} \\ \cdot \underline{y}_{s+1}^{(a_{s+1}(1), a_{s+1}(2), \dots, a_{s+1}(s))} \cdots \underline{y}_n^{(a_n(1), a_n(2), \dots, a_n(s))} \otimes v_\lambda.$$

Since $k_{s+1} \neq 0$, there exists some factor $\underline{y}_i^{\mathbf{a}_i} \neq 0$ with $s+1 \leq i \leq n$. Without loss of generality, we may assume that $\underline{y}_{s+1}^{\mathbf{a}_{s+1}} \neq 0$, then

$$x_{\alpha_{s+1}} \cdot \underline{y}_{s+1}^{(a_{s+1}(1), a_{s+1}(2), \dots, a_{s+1}(s))} \\ = \sum_{t=1}^s a_{s+1}(t) N_{\alpha_{s+1}, -(\alpha_t + \dots + \alpha_{s+1})} y_{\alpha_t + \dots + \alpha_s} \underline{y}_{s+1}^{(a_{s+1}(1), a_{s+1}(2), \dots, a_{s+1}(t)-1, \dots, a_{s+1}(s))} \\ + \underline{y}_{s+1}^{(a_{s+1}(1), a_{s+1}(2), \dots, a_{s+1}(s))} x_{\alpha_{s+1}}$$

where $N_{\alpha_{s+1}, -(\alpha_t + \dots + \alpha_{s+1})}$ is a structure constant of \mathfrak{g} relative to the Chevalley basis. Since $x_{\alpha_{s+1}}$ commutes with \underline{y}_t for any t with $t \neq s$, and annihilates v_λ , it follows

that

(3.5)

$$\begin{aligned} & x_{\alpha_{s+1}} \cdot u \otimes v_\lambda \\ &= \left(\sum_{\mathbf{a}} \sum_{t=1}^s l_{\mathbf{a}} a_{s+1}(t) N_{\alpha_{s+1}, -(\alpha_t + \dots + \alpha_{s+1})} \underline{y}_1^{a_1(1)} \dots \underline{y}_{s-1}^{(a_{s-1}(1), a_{s-1}(2), \dots, a_{s-1}(s))} y_{\alpha_t + \dots + \alpha_s} \right. \\ & \quad \left. \cdot \underline{y}_{s+1}^{(a_{s+1}(1), a_{s+1}(2), \dots, a_{s+1}(t)-1, \dots, a_{s+1}(s))} \dots \underline{y}_n^{(a_n(1), a_n(2), \dots, a_n(t), \dots, a_n(s))} \right) \otimes v_\lambda. \end{aligned}$$

Note that $\mathcal{Z}_P(\lambda)$ is free over $U_\chi(\mathfrak{u}_J^-)$. Since $u \otimes v_\lambda$ is a maximal weight vector, it follows from (3.5) and (3.1) that $u \otimes v_\lambda = 0$, a contradiction. Hence, $k_s > k_{s+1}$. \square

LEMMA 3.4. *Maintain the notations as in Lemma 3.3. Assume that*

$$\underline{y}_s^{\mathbf{m}_s} = y_{\alpha_1 + \alpha_2 + \dots + \alpha_s}^{m_s(1)} y_{\alpha_2 + \dots + \alpha_s}^{m_s(2)} \dots y_{\alpha_{s-1} + \alpha_s}^{m_s(s-1)} y_{\alpha_s}^{m_s(s)}$$

is a factor of one monomial summand of u such that $m_s(s)$ is maximal among all the powers of y_{α_s} in monomial summands of u . Then neither $y_{\alpha + \alpha_s}$ nor $y_{\alpha_s + \beta}$ with $\alpha \in R_I^+$, $\beta \in R_J^+$, appear in a monomial summand of u containing the factor $\underline{y}_s^{\mathbf{m}_s(s)}$.

PROOF. For the weight $\mu = \lambda - \sum_{i=1}^n k_i \alpha_i$ of $u \otimes v_\lambda$, we have $k_s > k_{s+1}$ by Lemma 3.3. Hence, there exists some nontrivial factor $\underline{y}_s^{\mathbf{a}_s}$ in each monomial summand of u .

(1) We first claim that there exists at least one monomial summand of u which contains a factor $y_{\alpha_s}^m$ with $m > 0$.

Otherwise, the factor $\underline{y}_s^{\mathbf{a}_s}$ of each monomial summand of u can be written as $\underline{y}_s^{\mathbf{a}_s} = y_{\alpha_1 + \dots + \alpha_s}^{a_s(1)} \dots y_{\alpha_{s-1} + \alpha_s}^{a_s(s-1)}$. Since $\mathbf{a}_s \neq \mathbf{0}$, there exist at least one $a_s(t) \neq 0$ for some $t \leq s-1$. Consider the action of $x_{\alpha_t + \dots + \alpha_{s-1}}$ on $u \otimes v_\lambda$. By assumption, there does not exist monomial summand of u which contains a factor $y_{\alpha_s}^m$ with $m > 0$. Since $x_{\alpha_t + \dots + \alpha_{s-1}} y_{\alpha_t + \dots + \alpha_s} = y_{\alpha_t + \dots + \alpha_s} x_{\alpha_t + \dots + \alpha_{s-1}} + N_{\alpha_t + \dots + \alpha_{s-1}, -(\alpha_t + \dots + \alpha_s)} y_{\alpha_s}$, it follows that $x_{\alpha_t + \dots + \alpha_{s-1}} \cdot u \otimes v_\lambda$ contains the following monomial summand

$$(3.6) \quad \underline{y}_1^{a_1(1)} \underline{y}_2^{a_2(1), a_2(2)} \dots \underline{y}_s^{(a_s(1), \dots, a_s(t)-1, \dots, a_s(s))} \dots \underline{y}_n^{(a_n(1), \dots, a_n(s))}$$

with $a_s(s) = 1$. Note that there does not exist a similar item as (3.6) among all the monomial summands of $x_{\alpha_t + \dots + \alpha_{s-1}} \cdot u \otimes v_\lambda$, then $x_{\alpha_t + \dots + \alpha_{s-1}} \cdot u \otimes v_\lambda \neq 0$. This is a contradiction. So, there exists at least one monomial summand of $u \otimes v_\lambda$ containing $y_{\alpha_s}^m$ with $m > 0$.

(2) Suppose that $y_{\alpha_t + \dots + \alpha_s}^{m_s(t)} \dots y_{\alpha_{s-1} + \alpha_s}^{m_s(s-1)} y_{\alpha_s}^{m_s(s)}$ with $m_s(t) \geq 1$ is a factor of a monomial summand of $u \otimes v_\lambda$. Consider the action of $x_{\alpha_t + \dots + \alpha_{s-1}}$ on $u \otimes v_\lambda$. A direct computation implies that $y_{\alpha_t + \dots + \alpha_s}^{m_s(t)-1} \dots y_{\alpha_{s-1} + \alpha_s}^{m_s(s-1)} y_{\alpha_s}^{m_s(s)+1}$ is a factor of a monomial summand of $x_{\alpha_t + \dots + \alpha_{s-1}} \cdot u \otimes v_\lambda$. Since $m_s(s)$ is maximal among all the powers of y_{α_s} in monomial summands of u , there does not exist a similar item as $y_{\alpha_t + \dots + \alpha_s}^{m_s(t)-1} \dots y_{\alpha_{s-1} + \alpha_s}^{m_s(s-1)-1} y_{\alpha_s}^{m_s(s)+1}$ among all the monomial summands of $x_{\alpha_t + \dots + \alpha_{s-1}} \cdot u \otimes v_\lambda$. Then $x_{\alpha_t + \dots + \alpha_{s-1}} \cdot u \otimes v_\lambda$ is not zero and this is contrary to the fact that $u \otimes v_\lambda$ is maximal.

So $y_{\alpha_t + \dots + \alpha_s}^{m_s(t)}$ with $m_s(t) \geq 1$ does not appear in the same monomial summand of u containing $y_{\alpha_s}^{m_s(s)}$, i.e., $y_{\alpha + \alpha_s}$ with $\alpha \in R_I^+$ do not appear in the monomial summand of u containing $y_{\alpha_s}^{m_s(s)}$. Similarly, $y_{\alpha_s + \beta}$ with $\beta \in R_J^+$ do not appear in the monomial summand of u containing $y_{\alpha_s}^{m_s(s)}$. \square

3.5. Proof of Claim 3.1. By Lemma 3.4, neither $y_{\alpha + \alpha_s}$, $\alpha \in R_I^+$ nor $y_{\alpha_s + \beta}$, $\beta \in R_J^+$ appear in the monomial summand of u provided it contains the factor $y_{\alpha_s}^{m_s(s)}$, where $m_s(s)$ is maximal.

Denote by u_s the sum of all those monomial summands of u containing $y_{\alpha_s}^{m_s(s)}$. By Lemma 3.4, u_s can be written as follows

$$(3.7) \quad u_s = \sum_{\mathbf{a}} l_{\mathbf{a}} \underline{y}_1^{a_1(1)} \underline{y}_2^{(a_2(1), a_2(2))} \dots \underline{y}_{s-1}^{(a_{s-1}(1), a_{s-1}(2), \dots, a_{s-1}(s-1))} y_{\alpha_s}^{m_s(s)} \\ \cdot \underline{y}_{s+1}^{(a_{s+1}(1), a_{s+1}(2), \dots, a_{s+1}(s-1))} \dots \underline{y}_n^{(a_n(1), a_n(2), \dots, a_n(s-1))}.$$

If $k_{s-1} \neq 0$, by a similar argument as in the proof of Lemma 3.4, there exists at least one monomial summand of u which contains $y_{\alpha_{s-1}}^m$ with $m > 0$. Let $m_{s-1}(s-1)$ be maximal among all those $a_{s-1}(s-1)$ appearing in (3.7). Then there exists a monomial summand of u_s with $a_{s-1}(s-1) = m_{s-1}(s-1)$ which does not contain $y_{\alpha_s + \beta}$ and $y_{\alpha + \alpha_s}$. By the same method as in the proof of Lemma 3.4, those $y_{\alpha + \alpha_{s-1}}$, $\alpha \in R^+$ and $y_{\alpha_{s-1} + \beta}$, $\beta \in R^+$ do not appear in the monomial summand of u_s in which $a_{s-1}(s-1) = m_{s-1}(s-1)$.

Denote by

$$u_{s-1} = \sum_{\mathbf{a}} l_{\mathbf{a}} \underline{y}_1^{a_1(1)} \underline{y}_2^{(a_2(1), a_2(2))} \dots \underline{y}_{s-2}^{(a_{s-2}(1), a_{s-2}(2), \dots, a_{s-2}(s-2))} y_{\alpha_{s-1}}^{m_{s-1}(s-1)} y_{\alpha_s}^{m_s(s)} \\ \cdot \underline{y}_{s+1}^{(a_{s+1}(1), a_{s+1}(2), \dots, a_{s+1}(s-2), 0, 0)} \dots \underline{y}_n^{(a_n(1), a_n(2), \dots, a_n(s-2), 0, \dots, 0)}.$$

Similar to the discussion above, if $k_{s-2} \neq 0$, then $y_{\alpha + \alpha_{s-2}}$, $\alpha \in R^+$ and $y_{\alpha_{s-2} + \beta}$, $\beta \in R^+$, do not appear in the monomial summand of u_{s-1} which contains $y_{\alpha_{s-2}}^{m_{s-2}(s-2)}$, where $m_{s-2}(s-2)$ is maximal among all those $a_{s-2}(s-2)$ appearing in (3.7).

Since I is connected, we can repeat the process above. Finally, we obtain that $u_1 = y_{\alpha_1}^{m_1(1)} y_{\alpha_2}^{m_2(2)} \dots y_{\alpha_{s-1}}^{m_{s-1}(s-1)} y_{\alpha_s}^{m_s(s)}$ with $m_s(s) > 0$ and $m_{s-i}(s-i) \geq 0$, $1 \leq i \leq s-1$. As u_1 is a summand of $u \otimes v_\lambda$, then $u \otimes v_\lambda \in \widehat{\mathcal{Z}}_P(\lambda)^{\lambda + \mathbb{Z}I} \cong_{\mathfrak{gl}} \widehat{L}_\chi(\lambda)^{\lambda + \mathbb{Z}I}$. This is contrary to the fact that $u \otimes v_\lambda$ generates the proper submodule of $\widehat{\mathcal{Z}}_P(\lambda)$. Therefore the Claim 3.1 holds in the case that $I = \{\alpha_1, \dots, \alpha_s\}$.

When $I = \{\alpha_t, \alpha_{t+1}, \dots, \alpha_n\}$, $1 < t \leq n$, the proof is similar. We complete the proof of Claim 3.1.

3.6. For the decomposition $\lambda = \lambda_0 + p\lambda_1$ with $\lambda_0 \in X'_1(T)$ and $\lambda_1 \in X(T)$, we have $\mathcal{F}(\widehat{\mathcal{Z}}_P(\lambda)) \cong \mathcal{Z}_P(\lambda_0)$ where \mathcal{F} is the forgetful functor (cf. [3, §11]). So the non-graded parabolic baby Verma module $\mathcal{Z}_P(\lambda_0)$ is also irreducible.

3.7. Assume that $I = \{\alpha_1, \alpha_2, \dots, \alpha_s\}$ with $s \leq n$. From the discussions above, if one replaces the condition $\lambda = (0, 0, \dots, 0)$ by $(m_1 - 1, \dots, m_s - 1, 0, \dots, 0)$ with $0 < m_i < p, \forall i$, Lemma 2.1 and Claim 3.1 still hold. We have the following consequence.

COROLLARY 3.5. Keep the same notations as above. Assume that I is connected in the Dynkin diagram with $\alpha_1 \in I$ (resp. $\alpha_n \in I$) and $\lambda_0 + \rho = (m_1, \dots, m_s, 1, \dots, 1)$ (resp. $\lambda_0 + \rho = (1, \dots, 1, m_t, \dots, m_n)$), $0 < m_i < p, \forall i$. Then $\widehat{\mathcal{Z}}_P(\lambda)$ is irreducible.

REMARK 3.6. For $\lambda = \lambda_0 + p\lambda_1$ with $\lambda_0 \in X'_1(T)$ and $\lambda_1 \in X(T)$, by Lemma 2.1, when λ_0 lies in the alcoves which contain the weight $(m_1, \dots, m_s, 1, \dots, 1) - \rho$ (resp. $(1, \dots, 1, m_t, \dots, m_n) - \rho$), $0 < m_i < p, \forall i$, the parabolic baby Verma module $\widehat{\mathcal{Z}}_P(\lambda)$ is irreducible.

4. Proof of Theorem 1.2 for types B_n, C_n and D_n

In this section, we give the proof of Theorem 1.2 for types B_n, C_n and D_n cases by cases.

4.1. Proof of Theorem 1.2 for type B_n . Let \mathfrak{g} be of type B_n and $\chi \in \mathfrak{g}^*$ be of standard Levi form with $I = \{\alpha \in \Pi \mid \chi(\mathfrak{g}_{-\alpha}) \neq 0\}$. Assume that $I = \{\alpha_s, \dots, \alpha_n\}$ for some $1 \leq s \leq n$ (note that α_n is a short root), i.e., we have the following Dynkin diagram

$$\circ - \dots - \circ - \underbrace{\bullet - \dots - \bullet}_{I} \Rightarrow \bullet$$

Fix an order of the Chevalley basis in \mathfrak{u}_J^- as follows

$y_{\alpha_n};$

$y_{\alpha_{n-1}+\alpha_n}, y_{\alpha_{n-1}+2\alpha_n}, y_{\alpha_{n-1}};$

\dots

$y_{\alpha_s+\alpha_{s+1}}, y_{\alpha_s+\alpha_{s+1}+\alpha_{s+2}}, \dots, y_{\alpha_s+2\alpha_{s+1}+\dots+2\alpha_{n-1}+2\alpha_n}, y_{\alpha_s};$

$y_{\alpha_{s-1}+\alpha_s}, y_{\alpha_{s-1}+\alpha_s+\alpha_{s+1}}, \dots, y_{\alpha_{s-1}+2\alpha_s+\dots+2\alpha_{n-1}+2\alpha_n};$

\dots

$y_{\alpha_1+\alpha_2+\dots+\alpha_s}, \dots, y_{\alpha_1+\alpha_2+\alpha_3+\dots+\alpha_n}, \dots, y_{\alpha_1+\alpha_2+2\alpha_3+\dots+2\alpha_n}, y_{\alpha_1+2\alpha_2+2\alpha_3+\dots+2\alpha_n}.$

Suppose that $u \otimes v_\lambda$ is a *maximal weight vector* of $\widehat{\mathcal{Z}}_P(\lambda)$ such that the submodule of $\widehat{\mathcal{Z}}_P(\lambda)$ generated by $u \otimes v_\lambda$ is proper. Then the weight μ of $u \otimes v_\lambda$ belongs to $W_p \cdot \lambda$ and can be written as $\mu = \lambda - \sum_{i=1}^n k_i \alpha_i$, $k_i \in \mathbb{Z}_+$ (cf. [4, Proposition 4.5]). We can use similar discussion as the case of type A_n . We enumerate the strategy as follows with the details omitted.

(i) Assume that $\lambda = (0, 0, \dots, 0)$. Then $k_s > 0$.

(ii) There exists $y_{\alpha_s}^m$ with $m > 0$ as a factor of some monomial summand of u . Assume that the power $m_s(s)$ of y_{α_s} is maximal among all the powers of y_{α_s} in monomial summands of u . First, we prove that $y_{\alpha_{s+1}+\alpha_s}$ do not appear in the same monomial summand of u which contains $y_{\alpha_s}^{m_s(s)}$. Next, we can prove that y_γ does not appear in the same monomial summand of u which contains $y_{\alpha_s}^{m_s(s)}$ as a factor, where $\gamma \in R^+$ and α_s is the first or last summand of γ .

(iii) u has a monomial summand $y_{\alpha_n}^{m_n(n)} y_{\alpha_{n-1}}^{m_{n-1}(n-1)} \cdots y_{\alpha_{s+1}}^{m_{s+1}(s+1)} y_{\alpha_s}^{m_s(s)}$.

Furthermore, we have

REMARK 4.1. Let $\lambda = \lambda_0 + p\lambda_1$ with $\lambda_0 \in X'_1(T)$ and $\lambda_1 \in X(T)$. Assume that λ_0 lies in the alcoves which contain the weight $(1, \dots, 1, m_s, \dots, m_n) - \rho$, $0 < m_i < p, \forall i$, then the parabolic baby Verma module $\widehat{\mathcal{Z}}_P(\lambda)$ is irreducible by Lemma 2.1.

4.2. Proof of Theorem 1.2 for type C_n . Let \mathfrak{g} be of type C_n and $\chi \in \mathfrak{g}^*$ be of standard Levi form with $I = \{\alpha \in \Pi \mid \chi(\mathfrak{g}_{-\alpha}) \neq 0\}$. Assume that $I = \{\alpha_1, \dots, \alpha_s\}, 1 \leq s \leq n$ (note that α_n is a long root), i.e., we have the following Dynkin diagram

$$\underbrace{\bullet - \bullet - \cdots - \bullet}_I - \circ - \cdots - \circ \Leftarrow \circ$$

Fix an order of the Chevalley basis in \mathfrak{u}_J^- as follows

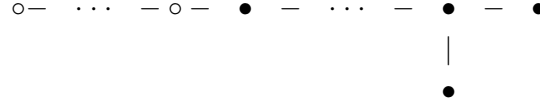
$$\begin{aligned} & y_{\alpha_1}; \\ & y_{\alpha_1+\alpha_2}, y_{\alpha_2}; \\ & \cdots \\ & y_{\alpha_{s-1}+\alpha_s}, y_{\alpha_{s-2}+\alpha_{s-1}+\alpha_s}, \cdots, y_{\alpha_1+\alpha_2+\cdots+\alpha_s}, y_{\alpha_s}; \\ & \cdots \\ & y_{\alpha_s+\alpha_{s+1}+\cdots+\alpha_{n-1}}, y_{\alpha_{s-1}+\alpha_s+\cdots+\alpha_{n-1}}, \cdots, y_{\alpha_1+\alpha_2+\cdots+\alpha_{n-1}}; \\ & y_{\alpha_s+\alpha_{s+1}+\cdots+\alpha_n}, \cdots, y_{\alpha_1+\alpha_2+\cdots+\alpha_n}, \cdots, y_{\alpha_1+2\alpha_2+\cdots+2\alpha_{n-1}+\alpha_n}, y_{2\alpha_1+2\alpha_2+\cdots+2\alpha_{n-1}+\alpha_n}. \end{aligned}$$

By a similar argument as the case of type B_n , we can prove Theorem 1.2 in the case of type C_n .

Furthermore, let $\lambda = \lambda_0 + p\lambda_1, \lambda_0 \in X'_1(T), \lambda_1 \in X(T)$. Assume that λ_0 lies in the alcoves which contain the weight $(m_1, \dots, m_s, 1, \dots, 1) - \rho$, $0 < m_i < p, \forall i$. Then the parabolic baby Verma module $\widehat{\mathcal{Z}}_P(\lambda)$ is irreducible by Lemma 2.1.

4.3. Proof of Theorem 1.2 for type D_n . Let \mathfrak{g} be of type D_n and $\chi \in \mathfrak{g}^*$ be of standard Levi form with $I = \{\alpha \in \Pi \mid \chi(\mathfrak{g}_{-\alpha}) \neq 0\}$.

(i) Assume that $I = \{\alpha_s, \dots, \alpha_{n-2}, \alpha_{n-1}, \alpha_n\}$, $1 \leq s \leq n-2$, i.e., we have the following Dynkin diagram



Fix an order of the Chevalley basis in \mathfrak{u}_J^- as follows

$$\begin{aligned} & y_{\alpha_n}, y_{\alpha_{n-1}}; \\ & y_{\alpha_n + \alpha_{n-2}}, y_{\alpha_{n-1} + \alpha_{n-2}}, y_{\alpha_n + \alpha_{n-1} + \alpha_{n-2}}, y_{\alpha_{n-2}}; \\ & \dots \\ & y_{\alpha_s + \alpha_{s+1}}, \dots, y_{\alpha_s + \dots + \alpha_{n-2} + \alpha_{n-1} + \alpha_n}, \dots, y_{\alpha_s + 2\alpha_{s+1} + 2\alpha_{s+2} + \dots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n}, y_{\alpha_s}; \\ & y_{\alpha_{s-1} + \alpha_s}, \dots, y_{\alpha_{s-1} + \dots + \alpha_{n-2} + \alpha_{n-1} + \alpha_n}, \dots, y_{\alpha_{s-1} + 2\alpha_s + 2\alpha_{s+1} + \dots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n}; \\ & \dots \\ & y_{\alpha_1 + \alpha_2 + \dots + \alpha_s}, \dots, y_{\alpha_1 + \dots + \alpha_{n-2} + \alpha_{n-1} + \alpha_n}, \dots, y_{\alpha_1 + 2\alpha_2 + 2\alpha_3 + \dots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n}. \end{aligned}$$

By a similar argument as the case of type B_n , we can prove Theorem 1.2 in this case for type D_n .

Furthermore, let $\lambda = \lambda_0 + p\lambda_1$ with $\lambda_0 \in X'_1(T)$ and $\lambda_1 \in X(T)$. Assume that λ_0 lies in the alcoves which contain the weight $(1, \dots, 1, m_s, \dots, m_n) - \rho$, $0 < m_i < p, \forall i$. Then the parabolic baby Verma module $\widehat{\mathcal{Z}}_P(\lambda)$ is irreducible by Lemma 2.1.

(ii) Assume that $I = \{\alpha_1, \dots, \alpha_{n-2}, \alpha_{n-1}\}$, i.e., we have the following Dynkin diagram



Fix an order of the Chevalley basis in \mathfrak{u}_J^- as follows

$$\begin{aligned} & y_{\alpha_1}; \\ & y_{\alpha_1 + \alpha_2}, y_{\alpha_2}; \\ & \dots \\ & y_{\alpha_{n-2} + \alpha_{n-1}}, y_{\alpha_{n-3} + \alpha_{n-2} + \alpha_{n-1}}, \dots, y_{\alpha_1 + \alpha_2 + \dots + \alpha_{n-1}}, y_{\alpha_{n-1}}; \\ & y_{\alpha_{n-2} + \alpha_n}, y_{\alpha_{n-2} + \alpha_{n-1} + \alpha_n}, \dots, y_{\alpha_1 + \dots + \alpha_{n-2} + \alpha_n}, y_{\alpha_1 + \alpha_2 + \dots + \alpha_{n-1} + \alpha_n}, \\ & y_{\alpha_1 + \dots + \alpha_{n-3} + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n}, \dots, y_{\alpha_1 + 2\alpha_2 + \dots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n}. \end{aligned}$$

By a similar argument as the case of type B_n , we can prove Theorem 1.2 in this case.

Furthermore, let $\lambda = \lambda_0 + p\lambda_1$ with $\lambda_0 \in X'_1(T)$ and $\lambda_1 \in X(T)$, and λ_0 lies in the alcoves which contain the weight $(m_1, \dots, m_{n-2}, m_{n-1}, 1) - \rho, 0 < m_i < p, \forall i$, then $\widehat{\mathcal{Z}}_P(\lambda)$ is irreducible by Lemma 2.1.

5. Proof of Corollary 1.3

5.1. Assume that \mathfrak{g} is of type A_n and $\chi \in \mathfrak{g}^*$ has standard Levi form associated with $I = \{\alpha \in \Pi \mid \chi(\mathfrak{g}_{-\alpha}) \neq 0\} = \{\alpha_1, \alpha_2, \dots, \alpha_{n-1}\}$. Set $\sigma = s_1 s_2 \cdots s_n$ where s_i is the simple reflection corresponding to α_i . Assume $\lambda_0 \in C_0$ with $\lambda_0 + \rho = (r_1, r_2, \dots, r_n)$. Then $0 \leq \sum_{i=1}^n r_i \leq p$.

Set $\lambda_i = \sigma^i \cdot \lambda_0$ for $1 \leq i \leq n$. Then $\lambda_i + \rho = (r_{n-i+2}, r_{n-i+3}, \dots, r_n, -(r_1 + \dots + r_n), r_1, r_2, \dots, r_{n-i})$. Each $\widehat{L}_\chi(w \cdot \lambda_0)$ with $w \in W$ is isomorphic to some $\widehat{L}_\chi(\lambda_i)$ with $0 \leq i \leq n$ (cf. [6, § 2.3]). Then $\{\widehat{L}_\chi(\lambda_i) \mid 0 \leq i \leq n\}$ is the set of isomorphism classes of simple modules in the block containing $\widehat{L}_\chi(\lambda_0)$.

For the decomposition $\lambda_i = \lambda_{i,0} + p\lambda_{i,1}$ with $\lambda_{i,0} \in X'_1(T)$ and $\lambda_{i,1} \in X(T)$, since $0 \leq \sum_{j=1}^n r_j \leq p$, we have $\lambda_{i,0} \in C_0, \forall i$. Therefore $\widehat{\mathcal{Z}}_P(\lambda_i)$ is irreducible by Theorem 1.2. Then $\widehat{\mathcal{Z}}_P(\lambda_i)$ has dimension $r_{n-i}p^{N-1}$, i.e., we get part (1) of Corollary 1.3 which coincides with [6, Theorem 2.6].

5.2. Assume that \mathfrak{g} is of type B_n and $\chi \in \mathfrak{g}^*$ has standard Levi form associated with $I = \{\alpha \in \Pi \mid \chi(\mathfrak{g}_{-\alpha}) \neq 0\} = \{\alpha_2, \alpha_3, \dots, \alpha_n\}$ (where α_n is a short root). Assume $\lambda_1 \in C_0$ with $\lambda_1 + \rho = (r_1, r_2, \dots, r_n)$. Then we have $0 \leq \sum_{i=1}^{n-1} 2r_i + r_n \leq p$. Let

$$w_i = \begin{cases} s_1 s_2 \cdots s_{i-1}, & \text{for } 1 \leq i \leq n, \\ s_1 s_2 \cdots s_n s_{n-1} \cdots s_{2n+1-i}, & \text{for } n+1 \leq i \leq 2n. \end{cases}$$

Set $\lambda_i = w_i \cdot \lambda_1$ for $1 \leq i \leq 2n$. Then $\{\widehat{L}_\chi(\lambda_i) \mid 1 \leq i \leq 2n\}$ is the set of isomorphism classes of simple modules in the block containing $\widehat{L}_\chi(\lambda_1)$ (cf. [6, § 3.8]).

Moreover, we have

$$\begin{aligned} \lambda_2 + \rho &= (-r_1, r_1 + r_2, r_3, \dots, r_{2n}), \\ \lambda_3 + \rho &= (-(r_1 + r_2), r_1, r_2 + r_3, \dots, r_{2n}), \\ &\dots\dots\dots, \\ \lambda_{2n} + \rho &= (-(r_1 + 2r_2 + 2r_3 + \dots + 2r_{n-1} + r_n), r_2, r_3, \dots, r_{2n}). \end{aligned}$$

Let

$$\lambda'_i = \begin{cases} s_2 s_3 \cdots s_i \cdot \lambda_i, & \text{for } 2 \leq i \leq n-1, \\ s_2 s_3 \cdots s_n s_{n-1} \cdots s_{2n+1-i} \cdot \lambda_i, & \text{for } n+1 \leq i \leq 2n-1. \end{cases}$$

Then

$$\begin{aligned}\lambda'_2 + \rho &= (r_2, -(r_1 + r_2), r_1 + r_2 + r_3, \dots, r_n), \\ \lambda'_3 + \rho &= (r_3, -(r_1 + r_2 + r_3), r_1, r_2 + r_3 + r_4, r_5, \dots, r_n), \\ &\dots\dots\dots, \\ \lambda'_{2n-1} + \rho &= (r_1, -(r_1 + r_2 + 2r_3 + \dots + 2r_{n-1} + r_n), r_3, \dots, r_n).\end{aligned}$$

It is obvious that the first component of $\lambda'_i + \rho$ is r_i for $1 \leq i \leq n-1$ and r_{2n-i} for $n+1 \leq i \leq 2n-1$. By [3, Proposition 11.9], we have $\widehat{L}_\chi(\lambda_i) \cong \widehat{L}_\chi(\lambda'_i)$ for $i \neq n, 2n$.

Since $0 \leq r_i \leq p$ and $0 \leq \sum_{i=1}^{n-1} 2r_i + r_n \leq p$, it follows from Remark 4.1 that $\widehat{\mathcal{Z}}_P(\lambda'_i)$ ($i \neq n, 2n$) is irreducible with dimension $r_i p^{N-1}$ for $1 \leq i \leq n-1$ and $r_{2n-i} p^{N-1}$ for $n+1 \leq i \leq 2n-1$, i.e., we get part (2) of Corollary 1.3 which coincides with [6, Proposition 3.13].

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